

7. Stochastic Orders

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So far: risk attitudes, i.e., patterns of individual behaviour in choices involving risk

This lecture: know how to rank lotteries/distributions in unambiguous manner among groups of individuals.

1. Rank distributions F and G s.t. every EU maximiser (with monotone u) would agree. (e.g., everyone would agree £2 for sure is better than £1 for sure)
2. Rank distributions according to 'riskiness', i.e., s.t. every risk-averse EU maximiser would agree.
(Is 2 stronger or weaker than 1?)

Overview

1. Stochastic Orders
2. Setup
3. First-Order Stochastic Dominance
4. Monotone Likelihood Ratio Order
5. Second-Order Stochastic Dominance
6. Background Risks
7. More

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Setup

- **Outcome space:** $X \subseteq \mathbb{R}$

$x \in X$: DM's final wealth.

- **Cumulative Probability Distributions Function** F

$F : \mathbb{R} \rightarrow [0, 1]$ s.t. F is nondecreasing, right-continuous, $\lim_{x \rightarrow -\infty} F(x) = 0$, and $\lim_{x \rightarrow \infty} F(x) = 1$ with support on X , i.e. $\mathbb{P}_F(X) = \int_X dF(x) = 1$.

Expectation Operator: $\mathbb{E}_F[\cdot]$

- $\mathcal{F}$: set of all cumulative probability functions on X

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First-Order Stochastic Dominance

Ranking of distributions s.t. every EU maximiser agrees:

Definition

A distribution F **first-order stochastically dominates** (FOSD) a distribution G , denoted by $F \geq_{FOSD} G$ iff, for all nondecreasing functions $u : X \rightarrow \mathbb{R}$, $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$.

Theorem

$\forall F, G \in \mathcal{F}, F \geq_{FOSD} G \iff \forall x \in X, F(x) \leq G(x).$

Remarkably simple characterisation of such strong property!

First-Order Stochastic Dominance

Theorem

$$\forall F, G \in \mathcal{F}, F \geq_{FOSD} G \iff \forall x \in X, F(x) \leq G(x).$$

Proof

$\implies$:

- $\forall a \in \mathbb{R}$, define $u_a(x) := \mathbf{1}_{\{x \geq a\}}$; $\mathbf{1}_A = 1$ if A is true, and 0 if ow.
- u_a nondecreasing $\forall a \in \mathbb{R}$.

$$F \geq_{FOSD} G \implies \mathbb{E}_F[u_a] \geq \mathbb{E}_G[u_a] \quad \forall a \in \mathbb{R}$$

$$\iff \int_X u_a(x) dF(x) \geq \int_X u_a(x) dG(x) \quad \forall a \in \mathbb{R}$$

$$\iff \int_{x \geq a} 1 dF(x) \geq \int_{x \geq a} 1 dG(x) \quad \forall a \in \mathbb{R}$$

$$\iff 1 - F(a) \geq 1 - G(a) \quad \forall a \in \mathbb{R}$$

$$\iff F(a) \leq G(a) \quad \forall a \in \mathbb{R}.$$

First-Order Stochastic Dominance

Theorem

$$\forall F, G \in \mathcal{F}, F \succeq_{FOSD} G \iff \forall x \in X, F(x) \leq G(x).$$

Proof

$\Leftarrow$: A small detour

We'll use a result in statistics called the inverse transform method.

Inverse Transform Sampling

Definition

$\forall F \in \mathcal{F}$, the **generalised inverse** (or **quantile function**) is given by $Q_F(\tau) := \min\{x \in \mathbb{R} \mid F(x) \geq \tau\}$, $\forall \tau \in (0, 1)$.

Proposition (Inverse Transform Sampling)

Let $F \in \mathcal{F}$ and $X \sim F$. Then, $X \stackrel{d}{=} Q_F(U)$, where $U \sim \text{Unif}(0, 1)$.

Simulating uniform rv is convenient and computationally efficient $\implies$ computationally efficient way of simulating *any* rv!

Inverse Transform Sampling

Definition

$$Q_F(\tau) := \min\{x \in \mathbb{R} \mid F(x) \geq \tau\}, \forall \tau \in (0, 1).$$

Proposition (Inverse Transform Sampling)

Let $F \in \mathcal{F}$ and $X \sim F$. Then, $X \stackrel{d}{=} Q_F(U)$, where $U \sim \text{Unif}(0, 1)$.

Proof

WTS $\mathbb{P}(Q_F(U) \leq x) = F(x)$.

(1) Q_F is nondecreasing:

$$F \text{ is nondecreasing} \implies \forall \tau' \geq \tau, \{x \in \mathbb{R} \mid F(x) \geq \tau'\} \subseteq \{x \in \mathbb{R} \mid F(x) \geq \tau\} \implies Q_F(\tau) \leq Q_F(\tau').$$

(2) $Q_F(F(x)) \leq x$: $\forall Q_F(F(x)) = \min\{y : F(y) = F(x)\}$ and $x \in \{y : F(y) = F(x)\}$.

(3) Take $\tau \in (0, 1), x \in \mathbb{R} : \tau < F(x)$. Then,

$$\tau < F(x) \implies Q_F(\tau) = \min\{y \in \mathbb{R} \mid F(y) \geq \tau\} \leq \min\{y \in \mathbb{R} \mid F(y) \geq F(x)\} = Q_F(F(x)) \leq x.$$

Inverse Transform Sampling

Definition

$$Q_F(\tau) := \min\{x \in \mathbb{R} \mid F(x) \geq \tau\}, \forall \tau \in (0, 1).$$

Proposition (Inverse Transform Sampling)

Let $F \in \mathcal{F}$ and $X \sim F$. Then, $X \stackrel{d}{=} Q_F(U)$, where $U \sim \text{Unif}(0, 1)$.

Proof

WTS $\mathbb{P}(Q_F(U) \leq x) = F(x)$.

(1) Q_F is nondecreasing. (2) $Q_F(F(x)) \leq x$. (3) $\tau < F(x) \implies Q_F(\tau) \leq Q_F(F(x)) \leq x$.

(4) As $Q_F(\tau) \leq x \implies \tau \leq F(x)$ (Q_F nondecreasing), then

$$\{U < F(x)\} \subseteq \{Q_F(U) \leq x\} \subseteq \{U \leq F(x)\}$$

$$\iff \mathbb{P}(U < F(x)) \leq \mathbb{P}(Q_F(U) \leq x) \leq \mathbb{P}(U \leq F(x))$$

$$\iff F(x) \leq \mathbb{P}(Q_F(U) \leq x) \leq F(x).$$

□

First-Order Stochastic Dominance

Theorem

$$\forall F, G \in \mathcal{F}, F \geq_{FOSD} G \iff \forall x \in X, F(x) \leq G(x).$$

Proof

$\Leftarrow$: Back to characterising FOSD. Fix u , define quantile functions Q_F and Q_G .

$$\begin{aligned} F(x) \leq G(x), \forall x \in X &\implies (F(x) \geq \tau \implies G(x) \geq \tau) \\ &\implies \{x \in X \mid F(x) \geq \tau\} \subseteq \{x \in X \mid G(x) \geq \tau\} \\ &\implies Q_F(\tau) \geq Q_G(\tau). \end{aligned}$$

$$\begin{aligned} F(x) \leq G(x), \forall x \in X &\implies Q_F(z) \geq Q_G(z), \forall z \in (0, 1) \\ &\implies u(Q_F(z)) \geq u(Q_G(z)), \forall z \in (0, 1) \quad \text{as } u \text{ nondec} \\ &\implies \int_{[0,1]} u(Q_F(z)) dz \geq \int_{[0,1]} u(Q_G(z)) dz \\ &\iff \int_X u(x) dF(x) \geq \int_X u(x) dG(x) \quad \text{inverse transform sampling} \\ &\iff \mathbb{E}_F[u] \geq \mathbb{E}_G[u]. \quad \square \end{aligned}$$

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Monotone Likelihood Ratio Order

Restrict attention to distrib. admitting either (i) density f or (ii) that have discrete support with pmf f

Definition

Let $F, G \in \mathcal{F}$ s.t. (i) either both admit a density, or (ii) both have discrete support. F **monotone likelihood ratio** dominates G ($F \geq_{MLR} G$) iff $f(x)/g(x)$ is nondecreasing in x .

Proposition

Let $F, G \in \mathcal{F}$ s.t. (i) either both admit a density, or (ii) both have discrete support. $F \geq_{MLR} G \implies F \geq_{FOSD} G$.

Proof

(1) $f(x)g(y) \geq f(y)g(x) \forall x \geq y \implies$ (a) $f(x)G(x) - F(x)g(x) \geq 0$ and

(b) $(1 - F(x))g(x) - f(x)(1 - G(x)) \geq 0 \forall x$.

(2) Note (a) $f(x)G(x) - F(x)g(x) \geq 0 \implies \frac{f(x)}{g(x)} \geq \frac{F(x)}{G(x)}$ and

(b) $(1 - F(x))g(x) - f(x)(1 - G(x)) \geq 0 \implies \frac{f(x)}{g(x)} \leq \frac{1 - F(x)}{1 - G(x)}$.

(a) and (b) $\implies \frac{1 - F(x)}{1 - G(x)} \geq \frac{F(x)}{G(x)} \iff (1 - F(x))G(x) \geq F(x)(1 - G(x)) \iff G(x) \geq F(x) \forall x$.

Proposition

Let $F, G \in \mathcal{F}$ s.t. (i) either both admit a density, or (ii) both have discrete support.
 $F \geq_{MLR} G \implies F \geq_{FOSD} G$.

MLR is in a sense a minimal condition so that FOSD is preserved under Bayesian updating – very convenient property (see exercise)

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 - Stochastic Orders in $\mathbb{R}^n$
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Second-Order Stochastic Dominance

Definition

For $F, G \in \mathcal{F}$, F **second-order stochastically dominates** (SOSD) G ($F \geq_{\text{SOSD}} G$) iff $\mathbb{E}_F[u] - \mathbb{E}_G[u] \geq 0$ for all nondecreasing, concave functions $u : \mathbb{R} \rightarrow \mathbb{R}$, such that $\mathbb{E}_F[u] - \mathbb{E}_G[u]$ is well-defined and $\int_{-\infty}^0 u(x)dF(x), \int_{-\infty}^0 u(x)dG(x) > -\infty$.

Restricting F, G to have bounded support, then $\int_{-\infty}^0 u(x)dF(x), \int_{-\infty}^0 u(x)dG(x) > -\infty$

Second-Order Stochastic Dominance

Definition

For $F, G \in \mathcal{F}$, F **second-order stochastically dominates** (SOSD) G ($F \geq_{SOSD} G$) $\iff$ $\mathbb{E}_F[u] - \mathbb{E}_G[u] \geq 0$ for all nondecreasing, concave functions $u : \mathbb{R} \rightarrow \mathbb{R}$, such that $\mathbb{E}_F[u] - \mathbb{E}_G[u]$ is well-defined and $\int_{-\infty}^0 u(x)dF(x), \int_{-\infty}^0 u(x)dG(x) > -\infty$.

WT find unambiguous ranking for risk averse DMs

Better understand riskiness

Separate individuals according to attitudes toward risk

How does $\geq_{SOSD}$ compare with $\geq_{FOSD}$? Which one allows for a finer comparison?

Which one is stronger?

Second-Order Stochastic Dominance

Theorem

$$\forall F, G \in \mathcal{F}', F \succeq_{\text{SOSD}} G \iff \forall x \in X, \int_{-\infty}^x F(s)ds \leq \int_{-\infty}^x G(s)ds.$$

Result has had troubled history; first version Hadar & Russell (1969 AER) and general version Tsefatian (1976 RES).

We'll prove the result for the subset of distributions with bounded support $\mathcal{F}'$.

Second-Order Stochastic Dominance

Theorem

$$\forall F, G \in \mathcal{F}', F \succeq_{\text{SOSD}} G \iff \forall x \in X, \int_{-\infty}^x F(s)ds \leq \int_{-\infty}^x G(s)ds.$$

Proof

Preliminaries: integration by parts: $\int_a^b u(x)dF(x) = F(b)u(b) - F(a)u(a) - \int_a^b F(x)du(x)$.

Bounded support $\implies$ choose $\bar{x}, \underline{x} : F(\underline{x}) = G(\underline{x}) = 0$ and $F(\bar{x}) = G(\bar{x}) = 1$

u defined on $(\underline{x} - \epsilon, \bar{x} + \epsilon)$.

Second-Order Stochastic Dominance

Theorem

$$\forall F, G \in \mathcal{F}', F \geq_{\text{SOSD}} G \iff \forall x \in X, \int_{-\infty}^x F(s)ds \leq \int_{-\infty}^x G(s)ds.$$

Proof

$\implies$:

- For $a \in \mathbb{R}$, define $u_a(x) := \mathbf{1}_{x \leq a}(x - a)$; nondecreasing and concave
- Integration by parts:

$$\begin{aligned} & \int_{\underline{x}}^a u_a(x) dF(x) - \int_{\underline{x}}^a u_a(x) dG(x) \\ &= (F(a) - G(a))(a - a) - (F(\underline{x}) - G(\underline{x}))u_a(\underline{x}) + \int_{\underline{x}}^a (G(x) - F(x))dx \\ &= \int_{\underline{x}}^a (G(x) - F(x)). \end{aligned}$$

Second-Order Stochastic Dominance

Theorem

$$\forall F, G \in \mathcal{F}', F \geq_{\text{SOSD}} G \iff \forall x \in X, \int_{-\infty}^x F(s)ds \leq \int_{-\infty}^x G(s)ds.$$

Proof

$\implies$:

- For $a \in \mathbb{R}$, define $u_a(x) := \mathbf{1}_{x \leq a}(x - a)$; nondecreasing and concave
- Integration by parts: $\int_{\underline{x}}^a u_a(x)dF(x) - \int_{\underline{x}}^a u_a(x)dG(x) = \int_{\underline{x}}^a (G(x) - F(x))dx$.

$$\begin{aligned} \mathbb{E}_F[u_a] - \mathbb{E}_G[u_a] \geq 0, \quad \forall a &\iff \int_{x \leq a} u_a(x)dF(x) \geq \int_{x \leq a} u_a(x)dG(x), \quad \forall a \\ &\iff \int_{x \leq a} u_a(x)dF(x) - \int_{x \leq a} u_a(x)dG(x) \geq 0, \quad \forall a \\ &\iff \int_{x \leq a} (G(x) - F(x))dx \geq 0, \quad \forall a \\ &\iff \int_{\underline{x}}^a F(x)dx \leq \int_{\underline{x}}^a G(x)dx, \quad \forall a. \end{aligned}$$

Second-Order Stochastic Dominance

Theorem

$$\forall F, G \in \mathcal{F}', F \geq_{\text{SOSD}} G \iff \forall x \in X, \int_{-\infty}^x F(s)ds \leq \int_{-\infty}^x G(s)ds.$$

Proof Sketch

$\Leftarrow$: Idea of the proof:

- (i) Fix u and do a nice linear interpolation u^n of u over an n -evenly-spaced-point grid on $[x, \bar{x}]$.
- (ii) Show that, for any n , we can express u^n as a finite sum of positive affine transformations of functions in the family u_a .
- (iii) Show that as $n \uparrow \infty$, u^n converges *uniformly* to u .
- (iv) Use (ii) to show that $\int_{-\infty}^x F(s)ds \leq \int_{-\infty}^x G(s)ds \forall x \implies \sum_{i=1}^{K_n} \mathbb{E}_F[u_{x_i^n}] - \mathbb{E}_G[u_{x_i^n}] \equiv \mathbb{E}_F[u^n] - \mathbb{E}_G[u^n] \geq 0 \forall n$.
- (v) Use (iii) and (iv) to show that $0 \leq \mathbb{E}_F[u^n] - \mathbb{E}_G[u^n] \rightarrow \mathbb{E}_F[u] - \mathbb{E}_G[u] \geq 0$.
(Filling in the blanks left as an exercise.)

Definition

For $F, G \in \mathcal{F}$, G is a **mean-preserving spread of** (MPS) F ($G \geq_{MPS} F$) iff $\exists$ random variables X, Y , and ϵ , such that $Y \stackrel{d}{=} X + \epsilon$, $X \sim F$, $Y \sim G$, and $\mathbb{E}[\epsilon \mid X] = 0$.

Properties of MPS

- (i) $G \geq_{MPS} F \implies F \geq_{SOSD} G$, but the converse is not true in general.
- (ii) $F \geq_{SOSD} G \implies \mathbb{E}_F[x] \geq \mathbb{E}_G[x]$.
- (iii) $G \geq_{MPS} F \implies \mathbb{E}_F[x] = \mathbb{E}_G[x]$ and $\mathbb{V}_F[x] \leq \mathbb{V}_G[x]$. (Prove it)
- (iv) $F \geq_{FOSD} G \implies F \geq_{SOSD} G$, but the converse is not true in general.
- (v) $\geq_{SOSD}$ and $\geq_{MPS}$ are partial orders.

Second-Order Stochastic Dominance in $\mathbb{R}^n$

Results extend to more general spaces.

Definition

For $F, G \in \Delta(\mathbb{R}^n)$. F is a **second-order stochastically dominates** (FOSD) G ($F \geq_{SOSD} G$) iff $\mathbb{E}_F[u] \geq \mathbb{E}_G[u]$ for all nondecreasing concave $u : \mathbb{R}^n \rightarrow \mathbb{R}$, whenever both expectations exist.

Theorem (Strassen 1965, Theorem 2.6.8)

Let F and G be distributions on $\mathbb{R}^n$ with bounded support. $F \geq_{SOSD} G$ if and only if $\exists X \sim F$ and $Y \sim G$ such that $X \geq \mathbb{E}[Y | X]$ a.s.

Result provides a way to define a joint distribution $H(x, y)$ such that the marginals over x and y equal F and G and $\int_{\mathbb{R}^n} yH(x, y)dy = \mathbb{E}[Y|X = x] \leq x$.

Mean-Preserving Spreads in $\mathbb{R}^n$

Corollary

Let F and G be distributions on $\mathbb{R}^n$ with bounded support. G is a mean-preserving spread of F if and only if $F \geq_{SOSD} G$ and $\mathbb{E}_F[x] = \mathbb{E}_G[x]$.

Why do we care? +Information $\implies$ MPS of beliefs!

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Background Risks

Often simplify comparison of lotteries $X \sim F$ and $Y \sim G$.

In reality: background risks ϵ ; right comparison is $X + \epsilon$ vs $Y + \epsilon$

When background risks are significant, it may overwhelm limited risk in X in Y

Pomatto, Strack, & Tamuz (2020 JPE): study connection between (independent) background risks and stochastic orders

Theorem

Let X and Y be random variables with finite variance.

- (i) If $\mathbb{E}[X] > \mathbb{E}[Y]$, then $\exists$ indep. random variable $\epsilon : X + \epsilon \geq_{FOSD} Y + \epsilon$.
- (ii) If $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\mathbb{V}[X] < \mathbb{V}[Y]$, then $\exists$ indep. random variable $\epsilon : X + \epsilon \geq_{SOSD} Y + \epsilon$.

Rationalises approximation of looking at expectation and variance in assessing assets when facing significant background noise.

Furthermore: if $\mathbb{E}[X] > \mathbb{E}[Y]$ and want everyone to prefer X to Y , can throw in some well calibrated background noise.

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More on Stochastic Orders

Meyer and Strulovici (2012): ordering interdependence

useful for finance (valuing portfolios), empirical work (inputting data), measuring alignment of preferences in decision-making in groups (e.g. voting), etc.

Kleiner, Moldovanu, and Strack (2021 Ecta) derive properties related to MPS and leverage these to study auctions, delegation, and decision-making under uncertainty (among others).

MCS with stochastic orders: distributional comparative statics in macro models, in games, in information, etc. (Jensen 2018 RES)

Reference textbooks: Muller and Stoyan (2002), Shaked and Shanthikumar (2007).